Identifying Groups in a Boolean Algebra

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Abstract

We study the problem of determining memberships to the groups in a Boolean algebra. The Boolean algebra is composed of basic groups (e.g., "J" and "K") and the other groups that are derived from basic groups through the conjunction, disjunction, or negation operations (e.g., "J and K", "J or K", "not J", etc.). All groups, basic and derived, are to be identified simultaneously based on the opinions of the potential members. Our main results are characterizations of (social decision) rules by means of independence axioms that are variants of Arrow's independence of irrelevant alternatives. We report that any of these independence axioms, together with other fairly mild axioms, implies simple decision schemes that focus on a single fixed vote for every membership decision. These rules are characterized earlier in the binary and multinary group identification models by Miller (2008) and Cho and Ju (2017). We unify these two models and their main results. Our extended setup uncovers implicit constraints in the earlier studies. Dropping these constraints, as we propose, leads to quite a rich spectrum of rules. Among them are the consent rules by Samet and Schmeidler (2003).

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1 Introduction

We investigate the problem of identifying members of a collective or a group based on individual opinions. This problem, called group identification, is formalized as a simple opinion aggregation problem by Kasher and Rubinstein (1997). Given a group, each person has an opinion on who belong to the group. Taking all persons' opinions as input, a social decision rule determines the membership for each person. The main goal is to come up with a proposal for well-behaved rules from normative ground. Group identification has recently attracted the attention of several authors; e.g., Samet and Schmeidler (2003), Sung and Dimitrov (2005), Houy (2007), Çengelci and Sanver (2010), and Ju (2010, 2013). All these authors study the binary model where there is only one group to be identified.

Miller (2008) and Cho and Ju (2017) consider group identification with two or more groups. In the presence of multiple groups, the issue of cross-group independence naturally arises: should the decision on one group rely on opinions on another? We formulate independence axioms similar to independence of irrelevant alternatives in Arrovian social choice theory (Arrow, 1951). Our results identify the rules that are characterized by independence axioms together with other relatively mild axioms. We unify the two models by Miller (2008) and Cho and Ju (2017) and their main characterizations. Our extended setup reveals implicit constraints in these two studies. Dropping the constraints, as we propose later, leads to quite a rich spectrum of rules. Among them are the "consent rules" by Samet and Schmeidler (2003).

Miller (2008) considers groups in a Boolean algebra: so his model has two or more basic groups (e.g., "J" and "K") and also derived groups defined by conjunction, disjunction, or negation of basic groups (e.g., "J and K", "J or K", "not J", and "J and not K"). Nevertheless, he treats the problems concerning all basic and derived groups as separate binary decision problems that are solved "independently". In other words, his model imposes the constraint of independent decision making across groups. The group-wise decisions are required to be "consistent" with respect to conjunction and disjunction[1]: the conjunction (disjunction) of the decision on group "J" and the deci-

[1]Once this requirement is met, the decisions for binary problems are consistent with respect to negation as well.

sion on group “K” should coincide with the decision on the derived group “J and K” (“J or K”). We find that in Miller (2008), this mild consistency axiom has a very strong implication mainly due to the constraint of independent decision making across groups.

The problem of identifying two or more groups is also studied by Cho and Ju (2017). They extend the binary model to a multinary model by simultaneously determining memberships to multiple groups. Crucial in their investigation is an independence axiom: the decision on each group should depend only on the opinions on that group, independently of the opinions on the other groups. This axiom is reminiscent of independence of irrelevant alternatives in Arrovian social choice theory (Arrow, 1951). A limitation of Cho and Ju (2017) is that each person is assumed to be a member of one and only group, so that the identified groups partition the set of all persons involved. We show that this partitioning (or single-membership) constraint is behind the strong implications of their independence axiom.

To subsume the models by Miller (2008) and Cho and Ju (2017), we consider groups in a Boolean algebra as in Miller (2008) and seek to determine memberships to all groups simultaneously as in Cho and Ju (2017). Yet we do not impose, in our model set-up, either independent decision making across groups as Miller (2008) does or the partitioning (single-membership) constraint as Cho and Ju (2017) do. Instead, we formulate the two properties as axioms in our model. This allows us to assess the exact roles they play in the two papers and to compare them on an equal footing.

In our model, a problem is defined as a list of binary (0 or 1) opinion matrices for all groups in a Boolean algebra. An opinion matrix for a group is composed of row vectors representing individual opinions on who belong to the group and who do not. A rule associates with each problem a list of binary decisions for all groups. Thus, a rule in our model can be viewed as a list of “component rules”, each of which identifies a group taking account of opinions on all groups.

The first independence axiom requires that the component rule for each “basic” group should be independent of opinions on the other “basic” groups. It is referred to as *independence of irrelevant opinions*. This is the main axiom in the reduced model of Cho and Ju (2017). The second independence axiom, called *component-wise independence*, requires that the component rule for each group, basic or derived, should be independent of opinions on the other groups. For example, the decision on group "J and K" should be independent, for example, of opinions on group "J" and opinions on group "K"; the decision on "J and K" should rely only on opinions on "J and K" (which are the conjunction of opinions on "J" and opinions on "K"). This axiom, obviously stronger than *independence of irrelevant opinions*, is implicitly assumed in the reduced framework of Miller (2008). The third independence axiom is intermediate in strength between *independence of irrelevant opinions* and *component-wise independence* and applies only to "complete" groups. A group is complete if it is the conjunction of qualification or disqualification for all basic groups. For instance, with two basic groups, "J" and "K", we have four complete groups: "J and K", "J and not K", "not J and K" and "not J and not K". *Complete group independence* requires that the decision on each complete group should be independent of opinions on the other complete groups. This is weaker than *component-wise independence*.

Our main results show that each of the three independence axioms, when combined with other fairly mild axioms, implies a simple decision scheme that relies only on a single fixed entry of an opinion matrix; that is, whether a person belongs to group "J", say, is determined by some entry (i, j) of the opinion matrix for group "J". These rules are called the one-vote rules.

Miller (2008) and Cho and Ju (2017) characterize the one-vote rules in two different models by means of different axioms, *consistency* in the former and *independence of irrelevant opinions* in the latter. Our extended setup subsumes the two models and allows us to draw a proper comparison of the two characterizations. Miller (2008) imposes the constraint of independent decisions across groups in the model, which amounts to *component-wise independence* in our model. Our results indicate that this constraint plays a more important role than *consistency* in Miller's (2008) characterization. In fact, even without consistency, we characterize rules exhibiting the feature of one-vote rules for each group; the critical one-votes across groups may vary inconsistently (so the family is larger than the family of one-vote rules).

On the other hand, Cho and Ju (2017) impose the partitioning property in their definition of rules. In our model, this property is explicitly formulated as an axiom, *unanimous basic group partitioning*. We find that the partitioning property in Cho and

Ju (2017) is crucial in characterizing the one-vote rules. This is because in the absence of *unanimous basic group partitioning*, a rich family of rules satisfy *independence of irrelevant opinions*, including the consent rules by Samet and Schmeidler (2003). Therefore, *independence of irrelevant opinions* without the partitioning property permits quite an optimistic outlook.

Finally, we adapt the consent rules to our extended setting and characterize them. Again, *independence of irrelevant opinions* plays a key role in our characterization. In the simpler model of Samet and Schmeidler (2003), the consent rules are characterized by three axioms: *monotonicity* (a rule should respond monotonically to changes in opinions), *person-by-person identification* (the membership decision for each person should be made based solely on the opinions about her)[2], and symmetry (names should not matter for membership decisions). In the extended setting, the latter axioms are satisfied by a number of rules other than the consent rules because they have no bite on the decisions across groups. Imposing, in addition, *independence of irrelevant opinions* and *consistency*, compels the relationship between group decisions to be more systematic and we obtain a characterization.

The rest of the paper is organized as follows. In Section 2, we construct the model and introduce axioms and examples of rules. Our results are in Section 3. Omitted proofs are in Section 4.

2 Model

We study the problem of determining membership for groups based on individual opinions. As mentioned in Section 1, this problem is studied in two models: the binary model (Miller, 2008) and the multinary model (Cho and Ju, 2017). In this section, we set up a model that subsumes the two and introduce axioms and definitions.

Let $\boldsymbol{N} \equiv \{1, \cdots, n\}$ be the set of persons and $\boldsymbol{G} \equiv \{1, \cdots, m\}$ the set of ***basic groups***. Assume that $n \geq 2$ and $m \geq 2$.[3] Basic groups are not necessarily mutually exclusive (i.e., a person can belong to two or more of them). Let $\mathcal{G}$ be the Boolean

[2]Samet and Schmeidler (2003) call this axiom "independence".

[3]If $m = 2$, with $G = \{k, \ell\}$, then we require $k \neq \neg\ell$, so that the Boolean algebra $\mathcal{G}$ (to be defined below) contains at least three non-trivial groups.

algebra generated by the set of basic groups G through conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$). Let $\mathbf{1} \in \mathcal{G}$ be the group "everyone" and $\mathbf{0} \in \mathcal{G}$ the group "no one". To avoid triviality, assume that for each $k \in G$, $k \neq \mathbf{1}$ and $k \neq \mathbf{0}$. We also assume, without loss of generality, that basic groups are not nested; i.e., there are no distinct $k, \ell \in G$ with $k \wedge \ell = k$.[4] We call the groups in $\mathcal{G} \backslash G$ ***derived groups***. A derived group is ***complete*** if it is a conjunction of qualification or disqualification for all basic groups in G. For example, when $G = \{a, b\}$, there are four complete groups: $a \wedge b$, $\neg a \wedge b$, $a \wedge \neg b$, and $\neg a \wedge \neg b$. Let $\mathcal{G}^c$ be the set of complete groups. Complete groups are mutually disjoint and they cover the whole set N of persons.

Let $g \in \mathcal{G}$. For each person $i \in N$, her ***g-opinion*** states who she thinks belong to group g. It is represented by a vector of 0's and 1's, $\mathbf{B}_i^g \equiv (\mathbf{B}_{ij}^g)_{j \in N} \in \{0,1\}^N$, where for all $j \in N$, $\mathbf{B}_{ij}^g = 1$ if and only if person i believes that person j is a member of group g. Stacking these opinions as rows, we obtain ***group g opinion matrix*** $\mathbf{B}^g$. Let $\mathcal{B} \equiv \{0,1\}^{N \times N}$ be the family of all these opinion matrices. A ***group g decision*** is a profile $x^g \in \{0,1\}^N$, where for all $i \in N$, $x_i^g = 1$ if and only if person i is a member of group g. Denote by $\mathbf{1}_{n \times n}$ and $\mathbf{1}_{1 \times n}$ the group g opinion matrix and the group g decision consisting of only 1's. The notation $\mathbf{0}_{n \times n}$ and $\mathbf{0}_{1 \times n}$ are defined similarly.

For all $g, g' \in \mathcal{G}$ and all $\mathbf{B}^g, \mathbf{B}^{g'} \in \mathcal{B}$, let $\mathbf{B}^g \wedge \mathbf{B}^{g'} \equiv \Big(\min\{\mathbf{B}_{ij}^g, \mathbf{B}_{ij}^{g'}\}\Big)_{i,j \in N}$ and $\mathbf{B}^g \vee \mathbf{B}^{g'} \equiv \Big(\max\{\mathbf{B}_{ij}^g, \mathbf{B}_{ij}^{g'}\}\Big)_{i,j \in N}$. Likewise, for all decisions $x^g, x^{g'} \in \{0,1\}^N$ for the two groups, let $x^g \wedge x^{g'} \equiv (\min\{x_i^g, x_i^{g'}\})_{i \in N}$ and $x^g \vee x^{g'} \equiv (\max\{x_i^g, x_i^{g'}\})_{i \in N}$. An (identification) ***problem*** is a profile $\mathbf{B} \equiv (\mathbf{B}^g)_{g \in \mathcal{G}} \in \mathcal{B}^{\mathcal{G}}$ of opinion matrices for all groups in $\mathcal{G}$ satisfying the following property: for all $g, g' \in \mathcal{G}$, $\mathbf{B}^{g \wedge g'} = \mathbf{B}^g \wedge \mathbf{B}^{g'}$, $\mathbf{B}^{g \vee g'} = \mathbf{B}^g \vee \mathbf{B}^{g'}$, $\mathbf{B}^{\mathbf{1}} = 1_{n \times n}$, and $\mathbf{B}^{\mathbf{0}} = 0_{n \times n}$. Call this property ***opinion-consistency***. It simply allows us to restrict attention to well-defined problems and the same is assumed in Miller (2008). Note that by *opinion-consistency*, for all $g \in \mathcal{G}$, $\mathbf{B}^{\neg g} = 1_{n \times n} - \mathbf{B}^g$.[5] Let $\mathfrak{B} \subsetneq \mathcal{B}^{\mathcal{G}}$ be the set of *opinion-consistent* problems. Consider a profile $(\mathbf{B}^k)_{k \in G}$ of opinion matrices for all basic groups. By *opinion-consistency*, the profile $(\mathbf{B}^k)_{k \in G}$

[4] Otherwise, for distinct $k, \ell \in G$ with $k \wedge \ell = k$, we can divide group ℓ into two subgroups k and $(\neg k) \wedge \ell$ and replace ℓ with the latter to avoid nestedness. Our basic groups are "atomic" groups obtained by iterating this process for all pairs of nested groups.

[5] It is evident that for all $\alpha, \beta \in \{0,1\}$, $\min\{\alpha, \beta\} = 0$ and $\max\{\alpha, \beta\} = 1$ imply $\alpha = 1 - \beta$. Since for all $g \in G$, $g \wedge \neg g = \mathbf{0}$ and $g \vee \neg g = \mathbf{1}$, $\mathbf{B}^g \wedge \mathbf{B}^{\neg g} = \mathbf{B}^{g \wedge \neg g} = \mathbf{B}^{\mathbf{0}} = 0_{n \times n}$ and $\mathbf{B}^g \vee \mathbf{B}^{\neg g} = \mathbf{B}^{g \vee \neg g} = \mathbf{B}^{\mathbf{1}} = 1_{n \times n}$. Therefore, $\mathbf{B}^{\neg g} = 1_{n \times n} - \mathbf{B}^g$.

generates a (unique) problem.

A ***decision*** is a profile $x \equiv (x^g)_{g\in\mathcal{G}} \in \{0,1\}^{N\times\mathcal{G}}$ of decisions for all groups in $\mathcal{G}$ with $x^{\mathbb{1}} = 1_{1\times n}$ and $x^{\mathbb{o}} = 0_{1\times n}$. It satisfies ***meet-consistency*** if for all $g, g' \in \mathcal{G}$, $x^{g\wedge g'} = x^g \wedge x^{g'}$. It satisfies ***join-consistency*** if for all $g, g' \in \mathcal{G}$, $x^{g\vee g'} = x^g \vee x^{g'}$. Note that the two consistency properties, together with $x^{\mathbb{o}} = 0_{1\times n}$ and $x^{\mathbb{1}} = 1_{1\times n}$, imply that for all $g \in \mathcal{G}$, $x^{\neg g} = 1_{1\times n} - x^g$.[6] Let $(x^k)_{k\in G}$ be a profile of decisions for all basic groups and call it a ***basic group decision***. Then $(x^k)_{k\in G}$ ***generates*** a (unique) decision satisfying the two *consistency* properties. A (social) ***rule*** $F : \mathfrak{B} \to \{0,1\}^{N\times\mathcal{G}}$ maps each problem into a decision. For all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, let $F^g(\mathbf{B})$ be the group g decision of F at $\mathbf{B}$.

Preliminary Axioms and Rules

Miller's axiom of consistency can be translated in thbe crucial e current setting to the requirement that the decisions across groups satisfy *meet-consistency* and *join-consistency.*

Consistency. For all $\mathbf{B} \in \mathfrak{B}$ and all $g, g' \in \mathcal{G}$,
(i) $F^{g\wedge g'}(\mathbf{B}) = F^g(\mathbf{B}) \wedge F^{g'}(\mathbf{B})$ (meet-consistency);
(ii) $F^{g\vee g'}(\mathbf{B}) = F^g(\mathbf{B}) \vee F^{g'}(\mathbf{B})$ (join-consistency).[7]

A simple way of constructing a *consistent* rule is first to define a decision rule for basic groups—let us call it a ***basic-group rule***—and then to extend the basic group decisions to a decision using *consistency.* For example, we may use the family of consent rules (Samet and Schmeidler, 2003) for basic-group rules. Let $(s_k, t_k)_{k\in G}$ be such that for all $k \in G$, $s_k, t_k \in \{1, \cdots, n, n+1\}$ and $s_k + t_k \le n+2$. The ***basic-group consent rule with*** $\boldsymbol{(s_k, t_k)_{k\in G}}$ satisfies the following: for all $\mathbf{B} \in \mathfrak{B}$, all $k \in G$, and all $i \in N$, (i) when $\mathbf{B}^k_{ii} = 1$, $F^k_i(\mathbf{B}) = 1$ if and only if $|\{j \in N : \mathbf{B}^k_{ji} = 1\}| \ge s_k$, and (ii) when $\mathbf{B}^k_{ii} = 0$, $F^k_i(\mathbf{B}) = 0$ if and only if $|\{j \in N : \mathbf{B}^k_{ji} = 0\}| \ge t_k$. A ***consent rule*** is the *consistent* extension of a basic-group consent rule. Thus, by construction, all consent rules are *consistent.*[8] The consent rule with $s_k = t_k = 1$ for all $k \in G$ is called the

[6] This can be explained as in Footnote 5.
[7] Miller (2008) calls these two properties meet separability and join separability.
[8] Alternatively, one may define a rule by applying conditions (i) and (ii) in the definition of the

liberal rule. A rule F is a ***one-vote rule*** if for all $i \in N$, there are $j, h \in N$ such that for all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, $F_i^g(\mathbf{B}) = \mathbf{B}_{jh}^g$. Clearly, all one-vote rules are *consistent*.

Finally we consider three "non-degeneracy" axioms. The first one is the strongest one.

Non-degeneracy. For all $g \in \mathcal{G} \backslash \{\mathfrak{1}, \mathfrak{0}\}$ and all $i \in N$, there are $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^g(\mathbf{B}) = 0$ and $F_i^g(\mathbf{B}') = 1$.

The next axiom requires non-degeneracy only for basic groups.

Basic Group Non-Degeneracy. For all $k \in G$ and all $i \in N$, there are $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^k(\mathbf{B}) = 0$ and $F_i^k(\mathbf{B}') = 1$.

The next axiom requires non-degeneracy only for complete groups.

Complete Group Non-Degeneracy. For all $g \in \mathcal{G}^c$ and all $i \in N$, there are $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ such that $F_i^g(\mathbf{B})$=0 and $F_i^g(\mathbf{B}') = 1$.

Partitioning Constraints

We next formulate as an axiom the partitioning constraint in the definition of rules by Cho and Ju (2017). It requires that each person belong to one and only one basic group, that is, the basic groups should partition the set of all persons.

Basic Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, $\big(\{i \in N : F_i^k(\mathbf{B}) = 1\}\big)_{k \in G}$ is a partition of N.

In the presence of *consistency*, this implies that for all distinct pairs of basic groups $k, \ell \in G$, $F^{k \wedge \ell}(\cdot)$ is degenerate, taking the constant value of $0_{1 \times n}$. This partitioning requirement may be too strong in our model and a weaker version may be formulated by requiring the partitioning property only when all persons agree with partitioning by basic groups.

Unanimous Basic Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, if each $i \in N$ partitions N into basic groups at $\mathbf{B}$ (i.e., for all $i \in N$, $(\{j \in N \colon \mathbf{B}_{ij}^k = 1\})_{k \in G}$ is a partition of N), then $\big(\{i \in N : F_i^k(\mathbf{B}) = 1\}\big)_{k \in G}$ is a partition of N.

basic-group consent rule to all groups $g \in \mathcal{G}$, basic or derived. However, this rule is not *consistent* unless $s = t = 1$, in which case, the rule is just the liberal rule.

Let $\mathfrak{B}^*$ be the restricted domain consisting of all problems $\mathbf{B} \in \mathfrak{B}$ such that $\sum_{k \in G} \mathbf{B}^k = 1_{n \times n}$. Then on the restricted domain, each person $i \in N$ believes that everyone is a member of one and only one group in G, that is, according to the opinion of each person $i \in N$, the basic groups $(\{j \in N : \mathbf{B}^k_{ij} = 1\})_{k \in G}$ partition the set of all persons. *Unanimous basic group partitioning* applies only to the problems in this restricted domain. In fact, the restricted domain coincides with the model studied by Cho and Ju (2017) where *unanimous basic group partitioning* is assumed in the definition of rules.

Consistency implies that any two complete groups are disjoint; hence complete groups necessarily partition the set of all persons.

Complete Group Partitioning. For all $\mathbf{B} \in \mathfrak{B}$, $\left(\{i \in N : F^g_i(\mathbf{B}) = 1\}\right)_{g \in \mathcal{G}^c}$ is a partition of N.

Independence Axioms

Three independence axioms are key in our investigation. The first independence axiom was introduced by Cho and Ju (2017). It says that the membership decision for a basic group k should rely only on opinions on that group, independently of the opinions on the other groups.

Independence of Irrelevant Opinions. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $k \in G$, if $\mathbf{B}^k = \mathbf{B}'^k$, then $F^k(\mathbf{B}) = F^k(\mathbf{B}')$.

This axiom does not require independence across derived groups, so the decision on $a \wedge b$ may depend on $\mathbf{B}^a$ and $\mathbf{B}^b$ as well as on $\mathbf{B}^{a \wedge b}$. Also, it is evident by definition that the one-vote rules and the consent rules satisfy *independence of irrelevant opinions.*

The next independence axiom imposes the same independence requirement not only for basic groups but also for derived groups. This is the constraint implicitly assumed in the binary model of Miller (2008).

Component-wise Independence. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $g \in \mathcal{G}$, if $\mathbf{B}^g = \mathbf{B}'^g$, then $F^g(\mathbf{B}) = F^g(\mathbf{B}')$.

This axiom requires independent decision making across derived groups as well as across basic groups. Thus, it is quite stronger than *independence of irrelevant opinions.* In fact, most consent rules violate it. For example, consider the "majority" consent rule with, for all $k \in G$, $s_k = t_k = \frac{n+1}{2}$. Then for all $i \in N$ and all $k \in G$, person i belongs to group k if and only if a majority believes her to be a member of group k, that is, $|\{j \in N \colon \mathbf{B}^k_{ji} = 1\}| \geq \frac{n+1}{2}$. Let $i \in N$. Consider $\mathbf{B} \in \mathfrak{B}$ such that (i) a majority identifies i as a member of group k; (ii) a majority identifies i as a member of group ℓ, and (iii) only a minority (less than $\frac{n+1}{2}$) identifies i as a member of group $k \wedge \ell$. Then for $\mathbf{B}$, the majority consent rule determines i as a member of groups k (by (i)), ℓ (by (ii)), and $k \wedge \ell$ (by *consistency*). Now consider $\mathbf{B}' \in \mathfrak{B}$ such that (i′) a minority identifies i as a member of group k; and (ii′) $\mathbf{B}'^{k \wedge \ell} = \mathbf{B}^{k \wedge \ell}$. For $\mathbf{B}'$, the majority consent rule determines i as a member of neither k (by (i′)) nor $k \wedge \ell$ (by *consistency*). Thus, although $\mathbf{B}$ and $\mathbf{B}'$ agree on the membership for group $k \wedge \ell$, the majority consent rule assigns different decisions to them, violating *component-wise independence*. It can be shown that among the consent rules, only the liberal rule is *component-wise independent*.

The following axiom, weaker than *component-wise independence*, requires decisions to be independent across complete groups.

Complete Group Independence. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ and all $g \in \mathcal{G}^c$, if $\mathbf{B}^g = \mathbf{B}'^g$, then $F^g(\mathbf{B}) = F^g(\mathbf{B}')$.

The consent rules are defined as a consistent extension of basic group consent rules. Alternatively, one may first use the consent rules for complete groups and then extend the complete group decisions to all other groups by *consistency*. Such rules satisfy *complete group independence*.

3 Results

Our first result is a characterization of rules satisfying *independence of irrelevant opinions*, together with *basic group non-degeneracy*, and *unanimous basic group partitioning*.

Theorem 1. (i) *If a rule F on $\mathfrak{B}^*$ (or $\mathfrak{B}$) satisfies* independence of irrelevant opinions*,* basic group non-degeneracy*, and* unanimous basic group partitioning*, then there is a one-vote rule $\hat{F}$ such that for all $\mathbf{B} \in \mathfrak{B}^*$ (or $\mathfrak{B}$) and all $k \in G$, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.*
(ii) *A rule F on $\mathfrak{B}^*$ (or $\mathfrak{B}$) satisfies* independence of irrelevant opinions*,* basic group

non-degeneracy*,* unanimous basic group partitioning*, and* consistency *if and only if it is a one-vote rule.*

Proof. The proof is in Section 4. □

Remark 1. Part (i) shows that even without *consistency*, we cannot get much far away from the one-vote rules. It also reveals that the family of rules characterized by Cho and Ju (2017) is similar to but larger than the family of one-vote rules characterized by Miller (2008).

On the other hand, if *independence of irrelevant opinions* is dropped in part (ii), one can find a rich family of rules, quite different from the one-vote rules, satisfying the other axioms. For example, fix a default group $\nu \in G$ and a profile of consent quotas $q \equiv (q_k)_{k \in G \backslash \{\nu\}}$ for basic groups other than ν, where for all $k \in G \backslash \{\nu\}$, $q_k \in \{1, \cdots, n\}$. Define the rule ${}^{q,\nu}F$ as follows: for all $\mathbf{B} \in \mathfrak{B}$ and all $i \in N$, (i) for all $k \in G \backslash \{\nu\}$, (i.a) if $\mathbf{B}^k_{ii} = 1$ and $|\{j \in N : \mathbf{B}^k_{ji} = 1\}| \geq q_k$, then ${}^{q,\nu}F^k_i(\mathbf{B}) = 1$, (i.b) otherwise, ${}^{q,\nu}F^k_i(\mathbf{B}) = 0$; (ii) ${}^{q,\nu}F^\nu_i(\mathbf{B}) = 1$ if and only if for all $k \in G \backslash \{\nu\}$, ${}^{q,\nu}F^k_i(\mathbf{B}) = 0$; and (iii) for all derived groups $g \in \mathcal{G} \backslash G$, ${}^{q,\nu}F^g_i(\mathbf{B})$ is the *consistent* extension of the basic group decisions $({}^{q,\nu}F^k(\mathbf{B}))_{k \in G}$. These rules satisfy all axioms in the theorem except for *independence of irrelevant opinions*.

To take another example, consider the *plurality rule* denoted PL and defined as follows: for all $\mathbf{B} \in \mathfrak{B}$, (i) for all $k \in G$ and all $i \in N$, $PL^k_i(\mathbf{B}) = 1$ if and only if for all $k' \in G$, $|\{j \in N : \mathbf{B}^k_{ji} = 1\}| \geq |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|$; (ii) for all derived groups $g \in \mathcal{G} \backslash G$, $PL^g(\mathbf{B})$ is the *consistent* extension of the basic group decisions $\left(PL^k(\mathbf{B})\right)_{k \in G}$. It is clear that this rule violates *independence of irrelevant alternatives* but satisfies *non-degeneracy*. However, there can be two or more basic groups to which person i belongs under PL. This can occur even for problems in $\mathfrak{B}^*$ and so PL violates *unanimous basic group partitioning*. However, we can define a "refinement" of PL using a linear ordering $\succ$ over basic groups, denoted ${}^{\succ}PL$, as follows: for all $\mathbf{B} \in \mathfrak{B}$, (i) for all $k \in G$ and all $i \in N$, ${}^{\succ}PL^k_i(\mathbf{B}) = 1$ if and only if for all $k' \in G \backslash \{k\}$, either $[|\{j \in N : \mathbf{B}^k_{ji} = 1\}| > |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|]$ or $[|\{j \in N : \mathbf{B}^k_{ji} = 1\}| = |\{j \in N : \mathbf{B}^{k'}_{ji} = 1\}|$ and $k' \succ k]$; (ii) for all derived groups $g \in \mathcal{G} \backslash G$, ${}^{\succ}PL^g(\mathbf{B})$ is the *consistent* extension of the basic group decisions $\left({}^{\succ}PL^k(\mathbf{B})\right)_{k \in G}$. Then this rule satisfies *unanimous basic group partitioning* too. We, therefore, find that in part (ii) of the theorem, *independence of irrelevant opinions* plays a much more significant role than *consistency*.

Also, some one-vote rules violate *basic group partitioning* on $\mathfrak{B}$. Thus, *unanimous basic group partitioning* in the theorem cannot be replaced by *basic group partitioning*.$\triangle$

Next, we derive an analogue of Miller's (2008) result in the current setup. When *component-wise independence* is added, a characterization of the one-vote rules obtains directly from his result.

Proposition 1 (Miller, 2008). *A rule on* $\mathfrak{B}$ *satisfies* component-wise independence, non-degeneracy, *and* consistency *if and only if it is a one-vote rule.*

Proof. To prove the nontrivial direction, let F be a rule satisfying the three axioms. Let $\phi\colon \mathcal{G} \times \mathcal{B} \to \{0,1\}^N$ be such that for all $g \in \mathcal{G}$ and all $B \in \mathcal{B}$, $\phi(g, B) \equiv F^g(\mathbf{B})$ for some $\mathbf{B} \in \mathfrak{B}$ with $\mathbf{B}^g = B$. Then by *component-wise independence*, ϕ is well-defined and for all $\mathbf{B} \in \mathfrak{B}$ and all $g \in \mathcal{G}$, $F^g(\mathbf{B}) = \phi(g, \mathbf{B}^g)$. Now applying Theorem 2.5 in Miller (2008) to $\phi(\cdot)$, we conclude that F is a one-vote rule. $\square$

In fact, we can obtain results that are stronger than Proposition 1 as corollaries to Theorem 1. Further, it turns out that the "decisive votes" that feature the one-vote rules emerge even in the absence of *consistency*.

Theorem 2. (i) *If a rule F on* $\mathfrak{B}$ *satisfies* complete group independence, complete group non-degeneracy, *and* complete group partitioning, *then there is a one-vote rule* $\hat{F}$ *such that for all* $\mathbf{B} \in \mathfrak{B}$ *and all* $g \in \mathcal{G}^c$, $F^g(\mathbf{B}) = \hat{F}^g(\mathbf{B})$.
(ii) *A rule F on* $\mathfrak{B}$ *satisfies* complete group independence, complete group non-degeneracy, *and* consistency *if and only if it is a one-vote rule.*

Proof. The proof is in Section 4. $\square$

Remark 2. This theorem can be used to prove Theorem 2.5 in Miller (2008) as follows. Consider a rule $\phi\colon \mathcal{G} \times \mathcal{B} \to \{0,1\}^N$ satisfying the axioms of *consistency* ("separability" as Miller calls it) and *non-degeneracy* in his theorem. Now use this rule ϕ to define the rule F such that for all $\mathbf{B} \in \mathfrak{B}$, all $g \in \mathcal{G}$, and all $i \in N$, $F_i^g(\mathbf{B}) \equiv \phi_i(g, \mathbf{B}^g)$. By construction, F satisfies *component-wise independence*. Also, *consistency* and *non-degeneracy* of ϕ imply *consistency* and *non-degeneracy* of F, respectively. Hence, part (ii) of the theorem implies that F is a one-vote rule, so that ϕ is a one-vote rule, as defined in the binary model. $\triangle$

Remark 3. Part (ii) of the theorem strengthens Proposition 1 by weakening *component-wise independence* to *complete group independence*. Also, part (i) indicates that even if *consistency* is dropped in part (ii) (while *complete group partitioning* is retained), we cannot get that far way from the one-vote rules. Any rule satisfying the other axioms coincides with a one-vote rule on its decisions for complete groups. However, if *complete group independence* is dropped, all consent rules satisfy the other axioms in part (ii). As in Remark 1, the independence axiom here, *complete group independence,* plays a much more significant role than *consistency*. $\triangle$

Note that on the restricted domain $\mathfrak{B}^*$, *independence of irrelevant opinions* is equivalent to *complete group independence*. Also, in the presence of the basic property "unanimity" (if $\mathbf{B}^k = 0_{n\times n}$, $F^k(\mathbf{B}) = 0_{1\times n}$), *unanimous basic group partitioning* is equivalent to *complete group partitioning*. Thus on the restricted domain $\mathfrak{B}^*$, we obtain almost the same results using the two sets of axioms.

It should be noted that both *component-wise independence* and *complete group independence* are quite more demanding than *independence of irrelevant opinions*. To explain this, suppose, for simplicity, that $G = \{a, b\}$. *Component-wise independence* and *complete group independence* require that for a decision on who are "a and b" ($a \wedge b$), we should only pay attention to opinions on who are "a and b" ($a \wedge b$) and ignore any other opinions, including the opinions on who are a and on who are b, which do not seem to be irrelevant to the decision on "a and b". For example, there are many cases of $\mathbf{B}^a$ and $\mathbf{B}^b$ that give rise to the same opinions on $a \wedge b$, $\mathbf{B}^{a\wedge b} = 0_{n\times n}$: $\mathbf{B}^a = 0_{n\times n}$ and $\mathbf{B}^b = 0_{n\times n}$, $\mathbf{B}^a = 1_{n\times n}$ and $\mathbf{B}^b = 0_{n\times n}$, and

$$\mathbf{B}^a = \begin{pmatrix} 1_{t\times t} & 0_{t\times(n-t)} \\ 0_{(n-t)\times t} & 1_{(n-t)\times(n-t)} \end{pmatrix} \text{ and } \mathbf{B}^b = \begin{pmatrix} 0_{t\times t} & 1_{t\times(n-t)} \\ 1_{(n-t)\times t} & 0_{(n-t)\times(n-t)} \end{pmatrix},$$

where $t \in \{1, \cdots, n-1\}$. The two independence axioms require that in all these different cases, there should be the same decision on $a \wedge b$, which seems too strong. Such an independence is not demanded by *independence of irrelevant opinions* since it requires independent decisions only across basic groups.

So far we have considered *independence of irrelevant opinions* and *unanimous basic group partitioning* at the same time. As Remark 1 notes, dropping *independence of irrelevant opinions* leads to a rich family of rules satisfying the other axioms in Theorem 1. We now drop *unanimous basic group partitioning.* Then all consent rules satisfy the other axioms in Theorem 1. They also satisfy the next three axioms, which adapt the main axioms in Samet and Schmeidler (2003) to our setting

Monotonicity. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$, if for all $k \in G$, $\mathbf{B}^k \leq \mathbf{B}'^k$, then for all $k \in G$, $F^k(\mathbf{B}) \leq F^k(\mathbf{B}')$. [9]

One may consider an alternative monotonicity axiom requiring that for all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$ with $\mathbf{B} \leq \mathbf{B}'$ (i.e., for all $g \in \mathcal{G}$, $\mathbf{B}^g \leq \mathbf{B}'^g$), $F(\mathbf{B}) \leq F(\mathbf{B}')$. Yet such an axiom is vacuous because by opinion-consistency, $\mathbf{B} \leq \mathbf{B}'$ implies $\mathbf{B} = \mathbf{B}'$ (if for any $g \in \mathcal{G}$, $\mathbf{B}^g \leq \mathbf{B}'^g$ and $\mathbf{B}^g \neq \mathbf{B}'^g$, then $\mathbf{B}^{\neg g} \geq \mathbf{B}'^{\neg g}$ and $\mathbf{B}^{\neg g} \neq \mathbf{B}'^{\neg g}$). For this reason, we restrict the scope of our monotonicity axiom to the basic groups. The next axiom says that to which groups a person belongs should depend only on the opinions about that person.

Person-by-Person Identification. For all $\mathbf{B}, \mathbf{B}' \in \mathfrak{B}$, if there is $j \in N$ such that for all $i \in N$ and all $g \in \mathcal{G}$, $\mathbf{B}^g_{ij} = \mathbf{B}'^g_{ij}$, then for all $g \in \mathcal{G}$, $F^g_j(\mathbf{B}) = F^g_j(\mathbf{B}')$.

This axiom corresponds to the "independence" axiom in Samet and Schmeidler (2003).

The next axiom says that the names of persons should not matter in identifying groups. Given $g \in \mathcal{G}$ and $\mathbf{B}^g \in \mathcal{B}$, for each permutation $\pi \colon N \to N$, let $\mathbf{B}^g_\pi$ be the group-g opinion matrix obtained from $\mathbf{B}^g$ permuting its entries through π; that is, for all $i, j \in N$, $(\mathbf{B}^g_\pi)_{ij} = \mathbf{B}^g_{\pi(i),\pi(j)}$. Let $\mathbf{B}_\pi \equiv (\mathbf{B}^g_\pi)_{g \in \mathcal{G}}$. Similarly, for each group-g decision x^g, let x^g_π be such that for all $i \in N$, $(x^g_\pi)_i = x^g_{\pi(i)}$. For each decision x, let $x_\pi \equiv (x^g_\pi)_{g \in \mathcal{G}}$.

Symmetry. For all $\mathbf{B} \in \mathfrak{B}$ and all permutations π on N, $F(\mathbf{B}_\pi) = F_\pi(\mathbf{B})$.

We now characterize the consent rules.

Theorem 3. *A rule on $\mathfrak{B}$ satisfies* independence of irrelevant opinions*,* consistency*,* monotonicity*,* person-by-person identification*, and* symmetry *if and only if it is a consent rule.*

[9] We write $\mathbf{B}^k \leq \mathbf{B}'^k$ if for all $i, j \in N$, $\mathbf{B}^k_{ij} \leq \mathbf{B}'^k_{ij}$; $F^k(\mathbf{B}) \leq F^k(\mathbf{B}')$ is similarly defined.

Proof. We only prove the "only if" part (the "if" part is clear). Let F be a rule on $\mathfrak{B}$ satisfying the five axioms. Let $k \in G$ be a basic group and define a group-k decision rule $f^k : \mathcal{B} \to \{0,1\}^N$ as follows: for all $\mathbf{B}^k \in \mathcal{B}$, $f(\mathbf{B}^k) = F^k(\mathbf{B}')$ for some $\mathbf{B}' \in \mathfrak{B}$ with $\mathbf{B}'^k = \mathbf{B}^k$. By *independence of irrelevant opinions*, f^k is well-defined. Then *monotonicity*, *person-by-person identification*, and *symmetry* of F imply that f^k satisfies the three corresponding axioms in Samet and Schmeidler (2003). Hence, their Theorem 1 implies that f^k is a group-k consent rule, represented by quotas (s_k, t_k). Repeating this argument for all basic groups, we obtain the basic-group consent rule $f \equiv (f_k)_{k \in G}$ with quotas $(s_k, t_k)_{k \in G}$. By construction, F coincides with f when restricted to the basic groups. Further, by *consistency*, F is the *consistent* extension of f. Thus, F is a consent rule. □

Remark 4. The axioms in the above theorem are independent, as shown by the following examples.

For a rule satisfying all but *independence of irrelevant opinions,* define F as follows: for all $\mathbf{B} \in \mathfrak{B}$ and all $i \in N$, (i) for all $k \in G$, $F_i^k(\mathbf{B}) = 1$ if and only if for all $\ell \in G$, $\mathbf{B}_{ii}^\ell = 1$; (ii) for all $g \in \mathcal{G} \backslash G$, $F_i^g(\mathbf{B})$ is a *consistent* extension of the basic group decisions $\left(F_i^k(\mathbf{B})\right)_{k \in G}$.

For a rule satisfying all but *consistency*, consider a rule that is similar to the consent rules but has consent quotas $(s_g, t_g)_{g \in \mathcal{G}}$ for all the derived groups as well as the basic groups. For instance, set $s_g = t_g = 2$ for all $g \in \mathcal{G}$.

For a rule satisfying all but *monotonicity*, modify the liberal rule to get F defined as follows: for all $\mathbf{B} \in \mathfrak{B}$ and all $i \in N$, (i) for all $k \in G$, $F_i^k(\mathbf{B}) = 1$ if and only if $\mathbf{B}_{ii}^k = 1$ and for some $j \in N$, $\mathbf{B}_{ji}^k = 0$; (ii) for all $g \in \mathcal{G} \backslash G$, $F_i^g(\mathbf{B})$ is a *consistent* extension of the basic group decisions $\left(F_i^k(\mathbf{B})\right)_{k \in G}$.

For a rule satisfying all but *person-by-person identification*, define F as follows: for all $\mathbf{B} \in \mathfrak{B}$ and all $i \in N$, (i) for all $k \in G$, $F_i^k(\mathbf{B}) = 1$ if and only if for all $j \in N$, $\mathbf{B}_{jj}^k = 1$; (ii) for all $g \in \mathcal{G} \backslash G$, $F_i^g(\mathbf{B})$ is a *consistent* extension of the basic group decisions $\left(F_i^k(\mathbf{B})\right)_{k \in G}$.

For a rule satisfying all but *symmetry*, consider dictatorial rules; e.g., the one-vote rule F such that for all $\mathbf{B} \in \mathfrak{B}$, all $i \in N$, and all $g \in \mathcal{G}$, $F_i^g(\mathbf{B}) = \mathbf{B}_{1i}^g$. △

4 Omitted Proofs

Our proofs rely on the characterization of the one-vote rules in the multinary model of Cho and Ju (2017). Thus, we briefly set up the multinary model and introduce some concepts and axioms.

Each person $i \in N$ has a ***multinary opinion*** $P_i \equiv (P_{ij})_{j \in N} \in G^N$, where for all $j \in N$ and all $k \in G$, $P_{ij} = k$ if person i believes that person j is a member of group k. A ***multinary problem*** is a profile $P \equiv (P_{ij})_{i,j \in N} \in G^{N \times N}$. Let $\mathcal{P}$ be the set of multinary problems. A ***multinary decision*** is a profile $x \equiv (x_i)_{i \in N} \in G^N$, where for all $i \in N$ and all $k \in G$, $x_i = k$ if person i is a member of group k. A ***multinary rule*** $f : \mathcal{P} \to G^N$ associates with each multinary problem a multinary decision. A multinary rule f is a ***one-vote rule*** if for all $i \in N$, there are $j, h \in N$ such that for all $P \in \mathcal{P}$, $f_i(P) = P_{jh}$.

Next, we state *independence of irrelevant opinions* and *non-degeneracy* in the multinary setup.

Independence of Irrelevant Opinions. Let $P, P' \in \mathcal{P}$ and $k \in G$. Suppose that for all $i, j \in N$, $P_{ij} = k$ if and only if $P'_{ij} = k$. Then for all $i \in N$, $f_i(P) = k$ if and only if $f_i(P') = k$.

Non-Degeneracy. For all $i \in N$, there are $P, P' \in \mathcal{P}$ such that $f_i(P) \neq f_i(P')$.

Cho and Ju (2017) show that in the multinary setup, the one-vote rules are characterized by *independence of irrelevant opinions* and *non-degeneracy*.

4.1 Proof of Theorem 1

We prove part (i) for the domain $\mathfrak{B}$ and skip the simpler proof for $\mathfrak{B}^*$. Let F be a rule on $\mathfrak{B}$ satisfying the three axioms. Let f be the multinary rule defined as follows using F: for all $P \in \mathcal{P}$, all $k \in G$, and all $i \in N$,

$$f_i(P) = k \Leftrightarrow F_i^k(\mathbf{B}) = 1 \; , \tag{1}$$

where $\mathbf{B} \in \mathfrak{B}$ is such that for all $\ell \in G$ and all $i, j \in N$, $\mathbf{B}_{ij}^{\ell} = 1$ if and only if $P_{ij} = \ell$. By *unanimous basic group partitioning*, f is well-defined. Since F satisfies *indepen-*

dence of irrelevant opinions and *basic group non-degeneracy*, f satisfies *independence of irrelevant opinions* and *non-degeneracy*, respectively. Thus, by Cho and Ju (2017), f is a one-vote rule; there is $h\colon N \to N \times N$ such that for all $P \in \mathcal{P}$, $f_i(P) = P_{h(i)}$. Let $\hat{F}$ be the one-vote rule such that for all $\mathbf{B} \in \mathfrak{B}$, all $g \in \mathcal{G}\backslash\{\mathfrak{1}, \mathfrak{o}\}$, and all $i \in N$, $\hat{F}_i^g(\mathbf{B}) = \mathbf{B}_{h(i)}^g$.

Let $\mathbf{B} \in \mathfrak{B}^*$. There is $P \in \mathcal{P}$ such that for all $k \in G$ and all $i, j \in N$, $\mathbf{B}_{ij}^k = 1$ if and only if $P_{ij} = k$. Since $f_i(P) = P_{h(i)}$, then by (1), for all $k \in G$, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.

Let $\mathbf{B} \in \mathfrak{B}\backslash\mathfrak{B}^*$. Let $k \in G$. There are opinion matrices $(\mathbf{B}'^{\ell})_{\ell \in G\backslash\{k\}}$ such that $\mathbf{B}^k + \sum_{\ell \in G\backslash\{k\}} \mathbf{B}'^{\ell} = 1_{n \times n}$. Then the profile $\big(\mathbf{B}^k, (\mathbf{B}'^{\ell})_{\ell \in G\backslash\{k\}}\big)$ of opinion matrices for all basic groups generates a problem in $\mathfrak{B}^*$. Denote by $\mathbf{B}''$ the problem so generated. By the argument in the previous paragraph, $F^k(\mathbf{B}'') = \hat{F}^k(\mathbf{B}'')$. Note that $\mathbf{B}''^k = \mathbf{B}^k$. Since both F and $\hat{F}$ are *independent of irrelevant opinions*, $F^k(\mathbf{B}) = \hat{F}^k(\mathbf{B})$.

Part (ii) follows directly from part (i) and *consistency*.

4.2 Proof of Theorem 2

Part (i). Let F be a rule satisfying the three axioms. Let $K \equiv \{1, \cdots, 2^m\}$ and fix a one-to-one correspondence $\theta\colon K \to \mathcal{G}^c$. For each $\mathbf{B} \in \mathfrak{B}$, by *opinion-consistency*, there is $P \in K^{N \times N}$ such that for all $i, j \in N$ and all $k \in K$,

$$P_{ij} = k \Leftrightarrow \mathbf{B}_{ij}^{\theta(k)} = 1 \; . \tag{2}$$

Conversely, for each $P \in K^{N \times N}$, there is $\mathbf{B} \in \mathfrak{B}$ satisfying (2). In fact, (2) defines a one-to-one correspondence $\Theta\colon K^{N \times N} \to \mathfrak{B}$. Let $f\colon K^{N \times N} \to K^N$ be defined as follows: for all $P \in K^{N \times N}$, all $i \in N$, and all $k \in K$, $f_i(P) = k$ if $F_i^{\theta(k)}(\Theta(P)) = 1$. By *complete group partitioning*, f is well-defined. *Complete group independence* of F implies *independence of irrelevant opinions* of f. *Complete group non-degeneracy* of F implies *non-degeneracy* of f. Finally, by Cho and Ju (2017), we conclude that f is a one-vote rule, which gives the desired conclusion.

Part (ii). To prove the non-trivial direction, let F be a rule satisfying *complete group independence*, *complete group non-degeneracy*, and *consistency*. Since *consistency* implies *complete group partitioning*, part (i) holds and the decision on any complete group by F coincides with the decision by a one-vote rule. Then by *consistency*, we conclude that the decision on any other group by F also coincides with the decision by the one-vote rule.

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